

Some new Fibonacci difference spaces of non-absolute type and compact operators

Anupam Das¹ and Bipan Hazarika^{1,*}

¹Department of Mathematics, Rajiv Gandhi University, Rono Hills, Doimukh-791 112,
Arunachal Pradesh, India

Email: anupam.das@rgu.ac.in; bh_rgu@yahoo.co.in

ABSTRACT. The aim of the paper is to introduced the spaces $c_0^\lambda(\hat{F})$ and $c^\lambda(\hat{F})$ which are the BK-spaces of non-absolute type and also derive some inclusion relations. Further, we determine the α -, β -, γ -duals of those spaces and also construct their bases. We also characterize some matrix classes on the spaces $c_0^\lambda(\hat{F})$ and $c^\lambda(\hat{F})$. Here we characterize the subclasses $\mathcal{K}(X, Y)$ of compact operators where X is $c_0^\lambda(\hat{F})$ or $c^\lambda(\hat{F})$ and Y is one of the spaces $c_0, c, l_\infty, l_1, bv$ by applying Hausdorff measure of noncompactness.

Key Words: Fibonacci numbers; α -, β -, γ -duals; Matrix Transformations; Measur of noncompactness; Hausdorff measure of noncompactness; Compact operator.

MSC: 11B39; 46A45; 46B45; 46B20.

1. INTRODUCTION

Let ω be the space of all real-valued sequences. Any vector subspace of ω is called a *sequence space*. By l_∞, c, c_0 , and l_p ($1 \leq p < \infty$), we denote the sets of all bounded, convergent, null sequences and p -absolutely convergent series, respectively. Also we use the conversions that $e = (1, 1, \dots)$ and $e^{(n)}$ is the sequence whose only non-zero term is 1 in the n th place for each $n \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$.

Let X and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk} , where $n, k \in \mathbb{N}$. We write $A = (a_{nk})$ instead of $A = (a_{nk})_{n,k=0}^\infty$. Then we say that A defines a matrix mapping from X into Y and we denote it by writing $A : X \rightarrow Y$ if for every sequence $x = (x_k)_{k=0}^\infty \in X$, the sequence $Ax = \{A_n(x)\}_{n=0}^\infty$, the A -transform of x , is in Y , where

$$(1.1) \quad A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k \quad (n \in \mathbb{N}).$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞ . Also, if $x \in \omega$, then we write $x = (x_k)_{k=0}^\infty$.

By (X, Y) , we denote the class of all matrices A such that $A : X \rightarrow Y$. Thus $A \in (X, Y)$ iff the series on the right-hand side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in X$ and we have $Ax \in Y$ for all $x \in X$.

The approach constructing a new sequence space by means of matrix domain has recently employed by several authors.

The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$(1.2) \quad X_A = \{x = (x_k) \in \omega : Ax \in X\}.$$

05-04-2016

*The corresponding author.

Kizmaz [12] introduced the notion of the difference operator Δ . The operator Δ denote the matrix $\Delta = (\Delta_{nk})$ defined by

$$(1.3) \quad \Delta_{nk} = \begin{cases} (-1)^{n-k}, & n-1 \leq k \leq n \\ 0, & 0 \leq k < n-1 \text{ or } k > n. \end{cases}$$

In the past, several authors studied matrix transformation on sequence spaces that are the matrix domain of the difference operator, or of the matrices of some classical methods of summability in different sequence spaces, for instance we refer to [6, 11, 14, 18] and references therein. Hausdorff measure of noncompactness of linear operators given by infinite matrices in some special classes of sequences spaces studied by [1, 16, 19, 21].

Define the sequence $\{f_n\}_{n=0}^{\infty}$ of Fibonacci numbers given by the linear recurrence relations $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}, n \geq 2$.

Fibonacci numbers have many interesting properties and applications. For example, the ratio sequences of Fibonacci numbers converges to the golden ratio which is important in sciences and arts. Also, some basic properties of Fibonacci numbers are given as follows:

$$(1.4) \quad \lim_{n \rightarrow \infty} \frac{f_{n+1}}{f_n} = \frac{1 + \sqrt{5}}{2} = \alpha \quad (\text{golden ratio}),$$

$$(1.5) \quad \sum_{k=0}^n f_k = f_{n+2} - 1 \quad (n \in \mathbb{N}),$$

$$(1.6) \quad \sum_k \frac{1}{f_k} \text{ converges ,}$$

$$(1.7) \quad f_{n-1}f_{n+1} - f_n^2 = (-1)^{n+1} \quad (n \geq 1) \quad (\text{Cassini formula})$$

Substituting for f_{n+1} in Cassini's formula yields $f_{n-1}^2 + f_n f_{n-1} - f_n^2 = (-1)^{n+1}$. For details see [13].

A sequence space X is called a FK -space if it is complete linear metric space with continuous coordinates $p_n : X \rightarrow \mathbb{R} (n \in \mathbb{N})$, where \mathbb{R} denotes the real field and $p_n(x) = x_n$ for all $x = (x_k) \in X$ and every $n \in \mathbb{N}$. A BK -space is a normed FK -space, that is a BK -space is a Banach space with continuous coordinates. The sapce $l_p (1 \leq p < \infty)$ is a BK -sapce with

$$\|x\|_p = \left(\sum_{k=0}^{\infty} |x_k|^p \right)^{1/p}$$

and c_0, c and l_{∞} are BK -spaces with

$$\|x\|_{\infty} = \sup_k |x_k|.$$

A sequence (b_n) in a normed space X is called a *Schauder basis* for X if every $x \in X$, there is a unique sequence (α_n) of scalars such that $x = \sum_n \alpha_n b_n$, i.e., $\lim_{m \rightarrow \infty} \|x - \sum_{n=0}^m \alpha_n b_n\| = 0$.

The α -, β -, γ -duals of the sequence space X are respectively defined by

$$X^{\alpha} = \{a = (a_k) \in \omega : ax = (a_k x_k) \in l_1 \forall x = (x_k) \in X\}.$$

$$X^{\beta} = \{a = (a_k) \in \omega : ax = (a_k x_k) \in cs \forall x = (x_k) \in X\},$$

and

$$X^{\gamma} = \{a = (a_k) \in \omega : ax = (a_k x_k) \in bs \forall x = (x_k) \in X\},$$

where cs and bs are the sequence spaces of all convergent and bounded series, respectively (See [2, 9, 20]).

$$\text{If } X \supset \phi \text{ is a } BK \text{ space and } a \in \omega \text{ we write } \|a\|_X^* = \sup \left\{ \left\| \sum_{k=0}^{\infty} a_k x_k \right\| : \|x\| = 1 \right\}.$$

Let X and Y be Banach spaces. A linear operator $L : X \rightarrow Y$ is called compact if its domain is all of X and for every bounded sequence $(x_n)_{n=0}^{\infty}$ in X , the sequence $(L(x_n))_{n=0}^{\infty}$ has a convergent subsequence in Y . We denote the class of such operators by $\mathcal{K}(X, Y)$.

Let us recall some definitions and well-known results.

Definition 1.1. Let (X, d) be a metric space, Q be a bounded subset of X and $B(x, r) = \{y \in X : d(x, y) < r\}$. Then the Hausdorff measure of noncompactness of Q , denoted by $\chi(Q)$, is defined by

$$\chi(Q) = \inf \left\{ \epsilon > 0 : Q \subset \bigcup_{i=1}^n B(x_i, r_i), x_i \in Q, r_i < \epsilon \quad (i = 1, 2, \dots, n), n \in \mathbb{N} \right\}.$$

Then the following results can be found in [3, 15].

If Q, Q_1 and Q_2 are bounded subsets of the metric space (X, d) , then we have $\chi(Q) = 0$ if and only if Q is totally bounded set,

$$\chi(Q) = \chi(\bar{Q}),$$

$$Q_1 \subset Q_2 \text{ implies } \chi(Q_1) \leq \chi(Q_2),$$

$$\chi(Q_1 \cup Q_2) = \max\{\chi(Q_1), \chi(Q_2)\}$$

and

$$\chi(Q_1 \cap Q_2) \leq \min\{\chi(Q_1), \chi(Q_2)\}.$$

If Q, Q_1 and Q_2 are bounded subsets of the normed space X , then we have

$$\chi(Q_1 + Q_2) \leq \chi(Q_1) + \chi(Q_2),$$

$$\chi(Q + x) = \chi(Q) \text{ for all } x \in X,$$

and

$$\chi(\lambda Q) = |\lambda| \chi(Q) \text{ for all } \lambda \in \mathbb{C}.$$

Definition 1.2. Let X and Y be Banach spaces and χ_1 and χ_2 be Hausdorff measures on X and Y . Then, the operator $L : X \rightarrow Y$ is called (χ_1, χ_2) -bounded if $L(Q)$ is bounded subset of Y for every subset Q of X and there exists a positive constant K such that $\chi_2(L(Q)) \leq K\chi_1(Q)$ for every bounded subset Q of X . If an operator L is (χ_1, χ_2) -bounded then the number $\|L\|_{(\chi_1, \chi_2)} = \inf \{K > 0 : \chi_2(L(Q)) \leq K\chi_1(Q) \text{ for all bounded } Q \subset X\}$ is called (χ_1, χ_2) -measure of noncompactness of L . In particular, if $\chi_1 = \chi_2 = \chi$, then we write $\|L\|_{(\chi, \chi)} = \|L\|_{\chi}$.

The idea of compact operators between Banach spaces is closely related to the Hausdorff measure of noncompactness, and it can be given as follows:

Let X and Y be Banach spaces and $L \in B(X, Y)$. Then the Hausdorff measure of noncompactness of L , is denoted by $\|L\|_{\chi}$, can be given by

$$(1.8) \quad \|L\|_{\chi} = \chi(L(S_X))$$

where $S_X = \{x \in X : \|x\| = 1\}$ and we have L is compact if and only if

$$(1.9) \quad \|L\|_{\chi} = 0$$

We also have

$$\|L\| = \sup_{x \in S_X} \|L(x)\|_Y.$$

2. THE SEQUENCE SPACES $c_0^\lambda(\hat{F})$ AND $c^\lambda(\hat{F})$ OF NON-ABSOLUTE TYPE

In this section, we introduce the spaces $c_0^\lambda(\hat{F})$ and $c^\lambda(\hat{F})$ and show that these spaces are the BK-spaces of non-absolute type which are linearly isomorphic to the spaces c_0 and c , respectively.

We shall assume throughout this paper that $\lambda = (\lambda_k)_{k=0}^{\infty}$ is strictly increasing sequence of positive reals tending to ∞ , that is $0 < \lambda_0 < \lambda_1 < \dots$ and $\lambda_k \rightarrow \infty$ as $k \rightarrow \infty$.

Recently, the sequence spaces c_0^λ and c^λ of non absolute type have been introduced by Mursaleen and Noman (see [22]) as follows:

$$c_0^\lambda = \left\{ x = (x_k) \in w : \lim_n \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k = 0 \right\}$$

and

$$c^\lambda = \left\{ x = (x_k) \in w : \lim_n \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k \text{ exists} \right\}.$$

Also, it has been shown that the inclusions $c_0 \subset c_0^\lambda$, $c \subset c^\lambda$ and $c_0^\lambda \subset c^\lambda$ hold.

Let f_n be the n th Fibonacci number for every $n \in \mathbb{N}$. The infinite matrix $\hat{F} = (f_{nk})$ was introduced by Kara [10] is defined as follows.

$$(2.1) \quad f_{nk} = \begin{cases} -\frac{f_{n+1}}{f_n}, & k = n-1 \\ \frac{f_n}{f_{n+1}}, & k = n \\ 0, & 0 \leq k < n-1 \text{ or } k > n \end{cases}$$

where $n, k \in \mathbb{N}$.

Define the sequence $y = (y_n)$, which will be frequently used, by the \hat{F} -transform of a sequence $x = (x_n)$, i.e., $y_n = \hat{F}_n(x)$, where

$$(2.2) \quad y_n = \begin{cases} \frac{f_0}{f_1} x_0 = x_0, & n = 0 \\ \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1}, & n \geq 1 \end{cases}$$

where $n \in \mathbb{N}$.

We employ a technique of obtaining a new sequence space by means of matrix domain. We thus introduce sequence spaces $c_0^\lambda(\hat{F})$ and $c^\lambda(\hat{F})$ are defined as follows.

$$c_0^\lambda(\hat{F}) = \left\{ x = (x_k) \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) \left(\frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right) = 0 \right\}$$

and

$$c^\lambda(\hat{F}) = \left\{ x = (x_k) \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) \left(\frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right) \text{ exists} \right\}.$$

We shall use the convention that any term with negative subscript is equal to zero, e.g. $\lambda_{-1} = 0$ and $x_{-1} = 0$.

We can redefine the spaces $c_0^\lambda(\hat{F})$ and $c^\lambda(\hat{F})$ by

$$(2.3) \quad c_0^\lambda(\hat{F}) = (c_0^\lambda)_{\hat{F}} \text{ and } c^\lambda(\hat{F}) = (c^\lambda)_{\hat{F}}.$$

It is immediate by (2.3) that the sets $c_0^\lambda(\hat{F})$ and $c^\lambda(\hat{F})$ are linear spaces with coordinate-wise addition and scalar multiplication. On the other hand, we define the matrix $\bar{F} = (\bar{f}_{nk})$ for all $n, k \in \mathbb{N}$ by

$$(2.4) \quad \bar{f}_{nk} = \begin{cases} \frac{1}{\lambda_n} \left[(\lambda_k - \lambda_{k-1}) \frac{f_k}{f_{k+1}} - (\lambda_{k+1} - \lambda_k) \frac{f_{k+2}}{f_{k+1}} \right], & k < n \\ \frac{1}{\lambda_n} (\lambda_n - \lambda_{n-1}) \frac{f_n}{f_{n+1}}, & k = n \\ 0, & k > n. \end{cases}$$

Then, it can be easily seen that

$$(2.5) \quad \bar{F}_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) \left(\frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right)$$

holds for all $n \in \mathbb{N}$ and $x = (x_k) \in w$, which leads us to the fact that

$$(2.6) \quad c_0^\lambda(\hat{F}) = (c_0)_{\bar{F}} \text{ and } c^\lambda(\hat{F}) = (c)_{\bar{F}}.$$

Moreover, it is obvious that \bar{F} is a triangle. Thus it has a unique inverse $\bar{F}^{-1} = (\bar{f}_{nk}^{-1})$ for all $n, k \in \mathbb{N}$ given by

$$(2.7) \quad \bar{f}_{nk}^{-1} = \begin{cases} \lambda_k f_{n+1}^2 \left[\frac{1}{\lambda_k - \lambda_{k-1}} \cdot \frac{1}{f_k f_{k+1}} - \frac{1}{\lambda_{k+1} - \lambda_k} \cdot \frac{1}{f_{k+1} f_{k+2}} \right] & , 0 \leq k < n \\ \lambda_n f_{n+1}^2 \cdot \frac{1}{\lambda_n - \lambda_{n-1}} \cdot \frac{1}{f_n f_{n+1}} & , k = n \\ 0 & , k > n. \end{cases}$$

Further, for any sequence $x = (x_k)$ we define the sequence $y = (y_k)$ such that $y = \bar{F}(x)$ and we observe that

$$(2.8) \quad y_k = \bar{F}_k(x) = \sum_{j=0}^{k-1} \frac{1}{\lambda_k} \left[(\lambda_j - \lambda_{j-1}) \frac{f_j}{f_{j+1}} - (\lambda_{j+1} - \lambda_j) \frac{f_{j+2}}{f_{j+1}} \right] x_j + \frac{1}{\lambda_k} (\lambda_k - \lambda_{k-1}) \frac{f_k}{f_{k+1}} x_k,$$

where $k \in \mathbb{N}$.

Now, we may begin with the following theorem which is essential in the text.

Theorem 2.1. *The sequence spaces $c_0^\lambda(\hat{F})$ and $c^\lambda(\hat{F})$ are BK-spaces with norm*

$$\|x\|_{c_0^\lambda(\hat{F})} = \|x\|_{c^\lambda(\hat{F})} = \|\bar{F}(x)\|_{l_\infty} = \sup_n |\bar{F}_n(x)|.$$

Proof. Since (2.6) holds and c_0 and c are BK-spaces with respect to their natural norm and the matrix \bar{F} is a triangle. Theorem 4.3.12 of Wilansky [25] gives the fact that $c_0^\lambda(\hat{F})$ and $c^\lambda(\hat{F})$ are BK-spaces with given norms. \square

Remark 2.2. *One can easily check that the absolute property is not satisfied by $c_0^\lambda(\hat{F})$ and $c^\lambda(\hat{F})$, that is $\|x\|_{c_0^\lambda(\hat{F})} \neq \||x\||_{c_0^\lambda(\hat{F})}$ and $\|x\|_{c^\lambda(\hat{F})} \neq \||x\||_{c^\lambda(\hat{F})}$. This shows that $c_0^\lambda(\hat{F})$ and $c^\lambda(\hat{F})$ are sequence spaces of non-absolute type, where $|x| = (|x_k|)$.*

Theorem 2.3. *The sequence spaces $c_0^\lambda(\hat{F})$ and $c^\lambda(\hat{F})$ of non-absolute type are linearly isomorphic to the spaces c_0 and c , respectively, that is $c_0^\lambda(\hat{F}) \cong c_0$ and $c^\lambda(\hat{F}) \cong c$.*

Proof. To prove this, we should show the existence of a linear bijection between the spaces $c_0^\lambda(\hat{F})$ and c_0 . Consider the transformation T defined, with the notation of (2.8), from $c_0^\lambda(\hat{F})$ to c_0 by $Tx = y = \bar{F}(x) \in c_0$ for every $x \in c_0^\lambda(\hat{F})$. Also, the linearity of T is clear. Further, it is trivial that $x = 0$ whenever $Tx = 0$ hence T is injective.

Further, let $y \in (y_k) \in c_0$ and we define the sequence $x = (x_k)$ by

$$(2.9) \quad x_k = \sum_{j=0}^k \sum_{i=j-1}^j (-1)^{j-i} \frac{\lambda_i y_i}{\lambda_j - \lambda_{j-1}} \cdot \frac{f_{k+1}^2}{f_j f_{j+1}}$$

for $k = 0, 1, 2, \dots$ and so on and $\bar{F}_n(x) = y_n$. This shows that $\bar{F}(x) = y$ and since $y \in c_0$, we obtain $\bar{F}(x) \in c_0$. Thus, we deduce that $x \in c_0^\lambda(\hat{F})$ and $Tx = y$. Hence T is surjective.

Moreover, for every $x \in c_0^\lambda(\hat{F})$ we have

$$\|Tx\|_{c_0} = \|Tx\|_{l_\infty} = \|y\|_{l_\infty} = \|\bar{F}(x)\|_{l_\infty} = \|x\|_{c_0^\lambda(\hat{F})}$$

which means that T is norm preserving. Consequently, T is a linear bijection which shows that $c_0^\lambda(\hat{F})$ and c_0 are linearly isomorphic.

Similarly, we can show that $c^\lambda(\hat{F}) \cong c$ and this concludes the proof. \square

Theorem 2.4. *The space l_∞ does not include the spaces $c_0^\lambda(\hat{F})$ and $c^\lambda(\hat{F})$.*

Proof. We have, from equation (2.5) that

$$\bar{F}_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) \left(\frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right) = \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) \hat{F}_k(x).$$

Let us define a sequence $x = (x_k) = (f_{k+1}^2)$. Since $f_{k+1}^2 \rightarrow \infty$ as $k \rightarrow \infty$ and $\hat{F}(x) = e^{(0)} = (1, 0, 0, \dots)$, therefore $\bar{F}_n(x) = \frac{\lambda_0}{\lambda_n} \rightarrow 0$ as $n \rightarrow \infty$.

Hence we can conclude that $x \in c_0^\lambda(\hat{F})$ but not in l_∞ . Similarly, we can show that $x \in c^\lambda(\hat{F})$ but not in l_∞ . \square

Theorem 2.5. *The inclusion $c_0^\lambda(\hat{F}) \subset c^\lambda(\hat{F})$ strictly holds.*

Proof. It is clear that $c_0^\lambda(\hat{F}) \subseteq c^\lambda(\hat{F})$. Consider the sequence $x = (x_n)$ defined by

$$(2.10) \quad x_n = \begin{cases} 1 & , n = 0 \\ f_{n+1}^2 \left(\sum_{j=1}^n \frac{1}{f_j f_{j+1}} + 1 \right) & , n \geq 1 \end{cases}$$

Then, we have $\bar{F}(x) = e$ and hence $\bar{F}(x) \in c \setminus c_0$ where $e = (1, 1, 1, \dots)$. Thus the sequence x in $c^\lambda(\hat{F})$ but not in $c_0^\lambda(\hat{F})$. Hence the inclusion $c_0^\lambda(\hat{F}) \subset c^\lambda(\hat{F})$ strict. \square

Theorem 2.6. *The inclusion $c \subset c^\lambda(\hat{F})$ and $c_0 \subset c_0^\lambda(\hat{F})$ strictly hold.*

Proof. Let $x = (x_k) \in c$. We have $c \subset c^\lambda$ and $c \subset c_0^\lambda$ if and only if $\liminf_{n \rightarrow \infty} \frac{\lambda_{n+1}}{\lambda_n} = 1$ (see [22]) and $\hat{F}(x) \in c$ as $\lim_{n \rightarrow \infty} \hat{F}_n(x)$ exists, therefore, $\hat{F}(x) \in c^\lambda$. This shows that $x \in c^\lambda(\hat{F})$.

Consequently, $c \subseteq c^\lambda(\hat{F})$.

Let $x = (x_k) = (f_{n+k}^2) \notin c$ but we have $\hat{F}(x) = (1, 0, 0, 0, \dots)$ so $y = (y_k)$, where $y_k = \bar{F}_k(x) = \frac{\lambda_0}{\lambda_k}$. Thus $y \in c$, hence $x \in c^\lambda(\hat{F})$ but not in c . We conclude that $c \subset c^\lambda(\hat{F})$ hold strictly.

Similarly, we can show that $c_0 \subset c_0^\lambda(\hat{F})$ hold strictly. \square

Now, because of the transformation T defined from $c_0^\lambda(\hat{F})$ to c_0 , is an isomorphism, the inverse image of the basis $\{e^{(k)}\}_{k=0}^\infty$ of the space c_0 is the basis for the new space $c_0^\lambda(\hat{F})$. Therefore, we have the following result.

Theorem 2.7. *Define the sequence $b^{(k)} = \{b_n^{(k)}\}_{n=0}^\infty$ for every fixed $k = 0, 1, 2, \dots$ by*

$$(2.11) \quad b_n^{(k)} = \begin{cases} 0 & , n < k \\ \frac{\lambda_k}{\lambda_k - \lambda_{k-1}} \cdot \frac{f_{n+1}^2}{f_k f_{k+1}} & , n = k \\ \frac{\lambda_k}{\lambda_k - \lambda_{k-1}} \cdot \frac{f_{n+1}^2}{f_k f_{k+1}} - \frac{\lambda_k}{\lambda_{k+1} - \lambda_k} \cdot \frac{f_{n+1}^2}{f_{k+1} f_{k+2}} & , n > k \end{cases}$$

where $n = 0, 1, 2, \dots$. Then the sequence $(b^{(k)})_{k=0}^\infty$ is a basis for the space $c_0^\lambda(\hat{F})$, and every $x \in c_0^\lambda(\hat{F})$ has a unique representation of the form

$$(2.12) \quad x = \sum_k \alpha_k b^{(k)}.$$

where $\alpha_k = \bar{F}_k(x)$ for all $k = 0, 1, 2, \dots$.

Proof. It is clear that the inclusion $\{b^{(k)}\} \subset c_0^\lambda(\hat{F})$ holds, since

$$(2.13) \quad \bar{F}(b^{(k)}) = e^{(k)} \in c_0, \quad k \in \mathbb{N}.$$

Let $x \in c_0^\lambda(\hat{F})$ be given. For every non-negative integer m , we put $x^{(m)} = \sum_{k=0}^m \alpha_k b^{(k)}$. Then

$$\text{we obtain by (2.13) that } \bar{F}(x^{(m)}) = \sum_{k=0}^m \alpha_k \bar{F}(b^{(k)}) = \sum_{k=0}^m \bar{F}_k(x) e^{(k)}$$

and hence

$$(2.14) \quad \bar{F}_n(x - x^{(m)}) = \begin{cases} 0, & 0 \leq n \leq m \\ \bar{F}_n(x), & n > m \end{cases}$$

where $n, m \in \mathbb{N}$.

Now, given $\epsilon > 0$, there exists a non-negative integer m_0 such that $|\bar{F}_m(x)| < \epsilon/2$ for all $m \geq m_0$.

Therefore, we have for every $m \geq m_0$ that

$$\|x - x^{(m)}\|_{c_0^\lambda(\hat{F})} = \sup_{n > m} |\bar{F}_n(x)| \leq \sup_{n > m_0} |\bar{F}_n(x)| \leq \frac{\epsilon}{2} < \epsilon.$$

which shows that $\lim_{m \rightarrow \infty} \|x - x^{(m)}\|_{c_0^\lambda(\hat{F})} = 0$ and hence x is represented as in (2.12).

If possible let there exists another representation

$$(2.15) \quad x = \sum_k \beta_k b^{(k)}.$$

Since $T \equiv \bar{F}$ is a linear transformation from $c_0^\lambda(\hat{F})$ to c_0 and is continuous, therefore we have $\bar{F}_n(x) = \sum_k \beta_k \bar{F}_n(b^{(k)}) = \beta_n$, for all $n = 0, 1, 2, \dots$. Thus we have $\alpha_k = \beta_k$ for all $k = 0, 1, 2, \dots$. Hence the representation (2.12) is unique. \square

Theorem 2.8. *The sequence $\{b, b^{(0)}, b^{(1)}, \dots\}$ is a basis for the space $c^\lambda(\hat{F})$ and every $x \in c^\lambda(\hat{F})$ has unique representation of the form,*

$$(2.16) \quad x = lb + \sum_k (\alpha_k - l) b^{(k)}$$

where $\alpha_k = \bar{F}_k(x)$ for all $k = 0, 1, 2, \dots$, the sequence $b = (b_n)$ is defined by

$$(2.17) \quad b_n = \begin{cases} 1, & n = 0 \\ f_{n+1}^2 \left(\sum_{j=1}^n \frac{1}{f_j f_{j+1}} + 1 \right), & n \geq 1, \end{cases}$$

the sequence $b^{(k)} = \{b_n^{(k)}\}_{n=0}^\infty$ is defined by (2.11) for every fixed $k = 0, 1, 2, \dots$ and

$$(2.18) \quad l = \lim_k \bar{F}_k(x).$$

Proof. Since $\{b^{(k)}\} \subset c_0^\lambda(\hat{F})$ and $\bar{F}(b) = e \in c$, the inclusion $\{b, b^{(k)}\} \subset c^\lambda(\hat{F})$ trivially holds. Further, let $x \in c^\lambda(\hat{F})$. Then there exists a unique l satisfying (2.18). Thus we have $y \in c_0^\lambda(\hat{F})$, where $y = x - lb$. Therefore, by Theorem 2.7 we have that the representation $y = \sum_k \beta_k b^{(k)}$ is unique, where $\beta_k = \bar{F}_k(x - lb) = \alpha_k - l$ for all k . Hence the representation (2.16) is unique. \square

Corollary 2.9. *The difference spaces $c_0^\lambda(\hat{F})$ and $c^\lambda(\hat{F})$ are separable.*

3. THE α -, β - AND γ -DUALS OF THE SPACES $c_0^\lambda(\hat{F})$ AND $c^\lambda(\hat{F})$

In this section, we determine the α -, β - and γ -duals of the sequence space $c_0^\lambda(\hat{F})$ and $c^\lambda(\hat{F})$ of non-absolute type.

We shall assume throughout our discussion that the sequences $x = (x_k)$ and $y = (y_k)$ are connected by the relation (2.8). Now we may begin with quoting the following lemmas (see [24]) which are needed to prove next theorems.

Lemma 3.1. $A \in (c_0 : l_1) = (c : l_1)$ if and only if

$$\sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} a_{nk} \right| < \infty.$$

Lemma 3.2. $A \in (c_0 : c)$ if and only if

$$(3.1) \quad \lim_n a_{nk} \text{ exists for each } k \in \mathbb{N},$$

$$(3.2) \quad \sup_n \sum_k |a_{nk}| < \infty$$

Lemma 3.3. $A \in (c : c)$ if and only if (3.1) and (3.2) hold, and

$$(3.3) \quad \lim_n \sum_k a_{nk} \text{ exists}.$$

Lemma 3.4. $A \in (c_0 : l_\infty) = (c : l_\infty)$ if and only if (3.2) holds.

Now, we prove the following results.

Theorem 3.5. The α -dual of the sequence space $c_0^\lambda(\hat{F})$ and $c^\lambda(\hat{F})$ is the set

$$b_1 = \left\{ a = (a_k) \in \omega : \sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} b_{nk} \right| < \infty \right\},$$

where the matrix $B = (b_{nk})$ is defined via the sequence $a = (a_n)$ by

$$b_{nk} = \begin{cases} \left(\frac{\lambda_k}{\lambda_k - \lambda_{k-1}} \cdot \frac{f_{n+1}^2}{f_k f_{k+1}} - \frac{\lambda_k}{\lambda_{k+1} - \lambda_k} \cdot \frac{f_{n+1}^2}{f_{k+1} f_{k+2}} \right) a_n & , k < n \\ \frac{\lambda_k}{\lambda_k - \lambda_{k-1}} \cdot \frac{f_{n+1}^2}{f_k f_{k+1}} \cdot a_n & , k = n \\ 0 & , k > n \end{cases}$$

for all $n, k \in \mathbb{N}$ and $a = (a_n) \in \omega$.

Proof. Let $a = (a_n) \in \omega$. Then by (2.8) and (2.9) we immediately derive that

$$(3.4) \quad a_n x_n = \sum_{k=0}^n \sum_{j=k-1}^k (-1)^{k-j} \frac{\lambda_j y_j}{\lambda_k - \lambda_{k-1}} \cdot \frac{f_{n+1}^2}{f_k f_{k+1}} a_n = B_n(y),$$

where $n = 0, 1, 2, \dots$. Thus we observed that by (3.4) that $ax = (a_n x_n) \in l_1$ when $x = (x_k) \in c_0^\lambda(\hat{F})$ or $c^\lambda(\hat{F})$ if and only if $By \in l_1$ when $y = (y_k) \in c_0$ or c i.e. $a = (a_n)$ is in the α -dual of the spaces $c_0^\lambda(\hat{F})$ or $c^\lambda(\hat{F})$ if and only if $B \in (c_0 : l_1) = (c : l_1)$. We, obtain by Lemma 3.1 that $a \in \left\{ c_0^\lambda(\hat{F}) \right\}^\alpha = \left\{ c^\lambda(\hat{F}) \right\}^\alpha$ iff

$$\sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} b_{nk} \right| < \infty$$

which gives $\left\{ c_0^\lambda(\hat{F}) \right\}^\alpha = \left\{ c^\lambda(\hat{F}) \right\}^\alpha = b_1$. □

Theorem 3.6. Define the sets b_2, b_3, b_4 and b_5 by

$$b_2 = \left\{ a = (a_k) \in \omega : \sum_{j=k}^{\infty} a_j f_{j+1}^2 \text{ exists for each } k \in \mathbb{N} \right\},$$

$$b_3 = \left\{ a = (a_k) \in \omega : \sup_n \sum_{k=0}^{n-1} |\bar{a}_k(n)| < \infty \right\},$$

$$b_4 = \left\{ a = (a_k) \in \omega : \sup_n \left| \frac{\lambda_n}{\lambda_n - \lambda_{n-1}} \cdot \frac{f_{n+1}^2}{f_n f_{n+1}} \cdot a_n \right| < \infty \right\}$$

and

$$b_5 = \left\{ a = (a_k) \in \omega : a_0 + \sum_{k=1}^{\infty} \left\{ f_{k+1}^2 \left(\sum_{j=1}^k \frac{1}{f_j f_{j+1}} + 1 \right) \text{ converges} \right\} \right\};$$

where

$$\bar{a}_k(n) = \lambda_k \left[\frac{a_k}{\lambda_k - \lambda_{k-1}} \cdot \frac{f_{k+1}^2}{f_k f_{k+1}} + \left(\frac{1}{\lambda_k - \lambda_{k-1}} \cdot \frac{1}{f_k f_{k+1}} - \frac{1}{\lambda_{k+1} - \lambda_k} \cdot \frac{1}{f_{k+1} f_{k+2}} \right) \sum_{j=k+1}^n f_{j+1}^2 a_j \right], \quad k < n.$$

Then $\left\{ c_0^\lambda(\hat{F}) \right\}^\beta = b_2 \cap b_3 \cap b_4$ and $\left\{ c^\lambda(\hat{F}) \right\}^\beta = b_3 \cap b_4 \cap b_5$.

Proof. Let $a = (a_k) \in \omega$ and consider the equality,

$$\begin{aligned} & \sum_{k=0}^n a_k x_k \\ &= \sum_{k=0}^n \left\{ \sum_{j=0}^k \left[\sum_{i=j-1}^j (-1)^{j-i} \frac{\lambda_i y_i}{\lambda_j - \lambda_{j-1}} \cdot \frac{f_{k+1}^2}{f_j f_{j+1}} \right] \right\} a_k \\ &= \sum_{k=0}^{n-1} \lambda_k \left[\frac{a_k}{\lambda_k - \lambda_{k-1}} \cdot \frac{f_{k+1}^2}{f_k f_{k+1}} + \left(\frac{1}{\lambda_k - \lambda_{k-1}} \cdot \frac{1}{f_k f_{k+1}} - \frac{1}{\lambda_{k+1} - \lambda_k} \cdot \frac{1}{f_{k+1} f_{k+2}} \right) \sum_{j=k+1}^n f_{j+1}^2 a_j \right] y_k + \frac{\lambda_n y_n}{\lambda_n - \lambda_{n-1}} \cdot \frac{f_{n+1}^2}{f_n f_{n+1}} a_n \\ &= \sum_{k=0}^{n-1} \bar{a}_k(n) y_k + \frac{\lambda_n y_n}{\lambda_n - \lambda_{n-1}} \cdot \frac{f_{n+1}^2}{f_n f_{n+1}} a_n \\ &= T_n(y); \text{ where } n = 0, 1, 2, 3, \dots \text{ and } T = (t_{nk}) \text{ is defined by} \end{aligned}$$

$$t_{nk} = \begin{cases} \bar{a}_k(n) & , k < n \\ \frac{\lambda_n}{\lambda_n - \lambda_{n-1}} \cdot \frac{f_{n+1}^2}{f_n f_{n+1}} \cdot a_n & , k = n \\ 0 & , k > n \end{cases}$$

for all $n, k \in \mathbb{N}$.

Then we have $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in c_0^\lambda(\hat{F})$ iff $Ty \in c$ whenever $y = (y_k) \in c_0$. Therefore $a = (a_k) \in \left\{ c_0^\lambda(\hat{F}) \right\}^\beta$ iff $T \in (c_0 : c)$. Therefore by using Lemma 3.2 we derive that

$$(3.5) \quad \sum_{j=k}^{\infty} a_j f_{j+1}^2 \text{ exists each } k = 0, 1, 2, \dots$$

$$(3.6) \quad \sup_n \sum_{k=0}^{n-1} |\bar{a}_k(n)| < \infty$$

and

$$(3.7) \quad \sup_n \left| \frac{\lambda_n}{\lambda_n - \lambda_{n-1}} \cdot \frac{f_{n+1}^2}{f_n f_{n+1}} \cdot a_n \right| < \infty.$$

Hence we conclude that $\left\{ c_0^\lambda(\hat{F}) \right\}^\beta = b_2 \cap b_3 \cap b_4$.

Similarly, from Lemma 3.3 we have $a = (a_k) \in \left\{ c^\lambda(\hat{F}) \right\}^\beta$ if and only if $T \in (c : c)$. Therefore, we derive from (3.1), (3.2) that (3.5), (3.6) and (3.7) hold.

Further, it can easily be seen that the equality

$$(3.8) \quad a_0 + \sum_{k=1}^n \left\{ f_{k+1}^2 \left(\sum_{j=1}^k \frac{1}{f_j f_{j+1}} + 1 \right) a_k \right\} = \sum_{k=0}^{n-1} \bar{a}_k(n) + \frac{\lambda_n}{\lambda_n - \lambda_{n-1}} \cdot \frac{f_{n+1}^2}{f_n f_{n+1}} a_n = \sum_k t_{nk}$$

where $n = 0, 1, 2, \dots$. Consequently, we obtain from Lemma 3.3 that

$$(3.9) \quad a_0 + \sum_{k=1}^n \left\{ f_{k+1}^2 \left(\sum_{j=1}^k \frac{1}{f_j f_{j+1}} + 1 \right) a_k \right\} \text{ converges.}$$

Thus $\sum_{j=k}^{\infty} a_j f_{j+1}^2$ exists is a weaker condition than $a_0 + \sum_{k=1}^n \left\{ f_{k+1}^2 \left(\sum_{j=1}^k \frac{1}{f_j f_{j+1}} + 1 \right) a_k \right\}$ converges. Hence we conclude that $\left\{ c^\lambda(\hat{F}) \right\}^\beta = b_3 \cap b_4 \cap b_5$. \square

Theorem 3.7. *The γ -dual of the spaces $c_0^\lambda(\hat{F})$ and $c^\lambda(\hat{F})$ is the set $b_3 \cap b_4$.*

Proof. This theorem can be proved similarly as the proof of the Theorem 3.6 by using Lemma 3.4. \square

4. SOME MATRIX MAPPINGS RELATED TO THE SEQUENCE SPACES $c_0^\lambda(\hat{F})$ AND $c^\lambda(\hat{F})$

In this section, we characterize the classes $(c^\lambda(\hat{F}) : l_p)$, $(c_0^\lambda(\hat{F}) : l_p)$, $(c^\lambda(\hat{F}) : c)$, $(c_0^\lambda(\hat{F}) : c)$, $(c^\lambda(\hat{F}) : c_0)$ and $(c_0^\lambda(\hat{F}) : c_0)$, where $1 \leq p \leq \infty$.

We assume that the sequences $x = (x_k)$ and $y = (y_k)$ are connected by $y = \bar{F}(x)$. Also for an infinite matrix $A = (a_{nk})$, we shall write that

$$\bar{a}_{nk}(m) = \lambda_k \left[\frac{a_{nk}}{\lambda_k - \lambda_{k-1}} \cdot \frac{f_{k+1}^2}{f_k f_{k+1}} + \left(\frac{1}{\lambda_k - \lambda_{k-1}} \cdot \frac{1}{f_k f_{k+1}} - \frac{1}{\lambda_{k+1} - \lambda_k} \cdot \frac{1}{f_{k+1} f_{k+2}} \right) \sum_{j=k+1}^m f_{j+1}^2 a_{nj} \right],$$

where $k < m$ and

$$\bar{a}_{nk} = \lambda_k \left[\frac{a_{nk}}{\lambda_k - \lambda_{k-1}} \cdot \frac{f_{k+1}^2}{f_k f_{k+1}} + \left(\frac{1}{\lambda_k - \lambda_{k-1}} \cdot \frac{1}{f_k f_{k+1}} - \frac{1}{\lambda_{k+1} - \lambda_k} \cdot \frac{1}{f_{k+1} f_{k+2}} \right) \sum_{j=k+1}^{\infty} f_{j+1}^2 a_{nj} \right]$$

for all $n, k, m \in \mathbb{N}$ provided the convergence of the series.

The following lemmas will be needed in our discussion.

Lemma 4.1. [25] *The matrix mapping between the BK-spaces are continuous.*

Lemma 4.2. [24] *$A \in (c : l_p)$ if and only if $\sup_{F \in \mathcal{F}} \sum_n \left| \sum_{k \in F} a_{nk} \right|^p < \infty$ where $1 \leq p < \infty$.*

Lemma 4.3. [24] *$A \in (c : c_0)$ if and only if*

$$(4.1) \quad \sup_n \sum_k |a_{nk}| < \infty,$$

$$(4.2) \quad \lim_n a_{nk} = 0 \text{ for all } k \in \mathbb{N},$$

$$(4.3) \quad \lim_n \sum_k a_{nk} = 0.$$

Lemma 4.4. [24] *$A \in (c_0, c_0)$ if and only if (4.1) and (4.2) hold.*

Now, we prove the following results.

Theorem 4.5. (i) *Let $1 \leq p < \infty$. Then $A \in (c^\lambda(\hat{F}) : l_p)$ if and only if*

$$(4.4) \quad \sup_{F \in \mathcal{F}} \sum_n \left| \sum_{k \in F} \bar{a}_{nk} \right|^p < \infty,$$

$$(4.5) \quad \sup_m \sum_{k=0}^{m-1} |\bar{a}_{nk}(m)| < \infty; (n \in \mathbb{N}),$$

$$(4.6) \quad a_{n0} + \sum_{k=1}^{\infty} \left\{ f_{k+1}^2 \left(\sum_{j=1}^k \frac{1}{f_j f_{j+1}} + 1 \right) a_{nk} \right\} \text{ converges, } n \in \mathbb{N},$$

$$(4.7) \quad \lim_{k \rightarrow \infty} \frac{f_{k+1}^2}{f_k f_{k+1}} \cdot \frac{\lambda_k}{\lambda_k - \lambda_{k-1}} a_{nk} = a_n; (n \in \mathbb{N}),$$

$$(4.8) \quad (a_n) \in l_p.$$

(ii) $A \in (c^\lambda(\hat{F}) : l_\infty)$ if and only if (4.6) and (4.7) hold, and

$$(4.9) \quad \sup_n \sum_k |\bar{a}_{nk}| < \infty,$$

$$(4.10) \quad (a_n) \in l_\infty.$$

Proof. Suppose that the conditions from (4.4) to (4.8) hold and let $x = (x_k) \in c^\lambda(\hat{F})$. Then we have $\{a_{nk}\}_{k \in \mathbb{N}} \in \{c^\lambda(\hat{F})\}^\beta$ for all $n \in \mathbb{N}$ and this implies that Ax exists. Also, it is clear that the associated sequence $y = (y_k)$ such that $T(y) = x$ in the space c and $y_k \rightarrow l$ as $k \rightarrow \infty$ for some suitable l . Combining the Lemma 4.2 with (4.4), we get $\bar{A} = (\bar{a}_{nk}) \in (c : l_p)$, $1 \leq p < \infty$.

Let us now consider the following equality derived by using the relation $y = \bar{F}(x)$ and the m th partial sum of the series $\sum_k a_{nk} x_k$ then we have

$$(4.11) \quad \sum_{k=0}^m a_{nk} x_k = \sum_{k=0}^{m-1} \bar{a}_{nk}(m) y_k + \frac{f_{m+1}^2}{f_m f_{m+1}} \cdot \frac{\lambda_m}{\lambda_m - \lambda_{m-1}} \cdot a_{nm} y_m$$

where $m, n \in \mathbb{N}$.

Since $y \in c$ and $\bar{A} \in (c : l_p)$; $\bar{A}y$ exists and so the series $\sum_k \bar{a}_{nk} y_k$ converges for all $n \in \mathbb{N}$.

From (4.6) we get $\sum_{j=k}^{\infty} a_{nj}$ converges for all $n, k \in \mathbb{N}$ and hence $\bar{a}_{nk}(m) \rightarrow \bar{a}_{nk}$ as $m \rightarrow \infty$.

Therefore, as $m \rightarrow \infty$ we get from (4.11) and (4.7) that

$$(4.12) \quad \sum_k a_{nk} x_k = \sum_k \bar{a}_{nk} y_k + l a_n \text{ for all } n \in \mathbb{N}$$

and which can be written as

$$(4.13) \quad A_n(x) = \bar{A}_n(y) + l a_n \text{ for all } n \in \mathbb{N}.$$

Therefore we have

$$(4.14) \quad \|A(x)\|_{l_p} \leq \|\bar{A}(y)\|_{l_p} + |l| \|(a_n)\|_{l_p} < \infty$$

which gives $Ax \in l_p$ and hence $A \in (c^\lambda(\hat{F}) : l_p)$, where $1 \leq p < \infty$.

Conversely suppose that $A \in (c^\lambda(\hat{F}) : l_p)$, where $1 \leq p < \infty$. Then $\{a_{nk}\} \in \{c^\lambda(\hat{F})\}^\beta$ for all $n \in \mathbb{N}$ and this with Theorem 3.6 implies the necessity conditions (4.5) and (4.6).

Since $c^\lambda(\hat{F})$ and l_p are BK -spaces, we have by Lemma 4.1 that there is a constant $M > 0$ such that

$$(4.15) \quad \|Ax\|_{l_p} \leq M \|x\|_{c^\lambda(\hat{F})}$$

for all $x \in c^\lambda(\hat{F})$. Now let $F \in \mathcal{F}$. Then the sequence $z = \sum_{k \in F} b^{(k)}$ is in $c^\lambda(\hat{F})$, where the sequence $b^{(k)} = \{b_n^{(k)}\}_{n \in \mathbb{N}}$ for every fixed $k \in \mathbb{N}$. We have by (2.13) that

$$\|z\|_{c^\lambda(\hat{F})} = \|\bar{F}(z)\|_{l_\infty} = \left\| \sum_{k \in F} \bar{F}(b^{(k)}) \right\|_{l_\infty} = \left\| \sum_{k \in F} e^{(k)} \right\|_{l_\infty} = 1.$$

Again for every $n \in \mathbb{N}$, we have $A_n(z) = \sum_{k \in F} A_n(b^{(k)}) = \sum_{k \in F} \sum_j a_{nj} b_j^{(k)} = \sum_{k \in F} \bar{a}_{nk}$. Since the inequality (4.15) is satisfied for the sequence $z \in c^\lambda(\hat{F})$, we have for any $F \in \mathcal{F}$ that $\left(\sum_n \left| \sum_{k \in F} \bar{a}_{nk} \right|^p \right)^{1/p} \leq M$ which implies $\sup_{F \in \mathcal{F}} \sum_n \left| \sum_{k \in F} \bar{a}_{nk} \right|^p < \infty$. Thus it follows by Lemma 4.2 that $\bar{A} = (\bar{a}_{nk}) \in (c : l_p)$.

Now let $y = (y_k) \in c \setminus c_0$ and consider the sequence $x = (x_k)$ defined by (2.9) for every $k \in \mathbb{N}$. Then $x \in c^\lambda(\hat{F})$ such that $y = \bar{F}(x)$. Therefore Ax and $\bar{A}y$ exist. This gives the series $\sum_k a_{nk}x_k$ and $\sum_k \bar{a}_{nk}y_k$ converges for all $n \in \mathbb{N}$.

We also have

$$\lim_m \sum_{k=0}^{m-1} \bar{a}_{nk}(m)y_k = \sum_k \bar{a}_{nk}y_k; (n \in \mathbb{N}).$$

As $m \rightarrow \infty$ from (4.11), we get $\lim_m \frac{f_{m+1}^2}{f_{m+1}f_m} \cdot \frac{\lambda_m}{\lambda_m - \lambda_{m-1}} \cdot a_{nm}y_m$ exists ($n \in \mathbb{N}$) and since $y \in c \setminus c_0$, we have $\lim_m \frac{f_{m+1}^2}{f_{m+1}f_m} \cdot \frac{\lambda_m}{\lambda_m - \lambda_{m-1}} \cdot a_{nm}$ exists; ($n \in \mathbb{N}$) which shows the necessity of (4.7) holds, where $l = \lim_k y_k$.

Again since $Ax \in l_p$ and $\bar{A}x \in l_p$ implies $\{a_n\} \in l_p$ by (4.13).

This completes the proof of part (i).

Part (ii) can be proved in the similar way of that used in the proof of part (i) by using Lemma 3.4. \square

Theorem 4.6. (i) Let $1 \leq p < \infty$. Then $A \in (c_0^\lambda(\hat{F}) : l_p)$ if and only if (4.4) and (4.5) hold, and

$$(4.16) \quad \sum_{j=k}^{\infty} a_{nj} \text{ exists } (n, k \in \mathbb{N}),$$

$$(4.17) \quad \left\{ \frac{f_{k+1}^2}{f_k f_{k+1}} \cdot \frac{\lambda_k}{\lambda_k - \lambda_{k-1}} \cdot a_{nk} \right\}_{k=0}^{\infty} \in l_\infty \quad (n \in \mathbb{N}).$$

(ii) $A \in (c_0^\lambda(\hat{F}) : l_\infty)$ if and only if (4.9), (4.16) and (4.17) hold

Proof. The proof is similar to the proof of Theorem 4.5. \square

Theorem 4.7. $A \in (c^\lambda(\hat{F}) : c)$ if and only if (4.6), (4.7), (4.9) hold and

$$(4.18) \quad \lim_n a_n = a,$$

$$(4.19) \quad \lim_n \bar{a}_{nk} = \alpha_k \quad (k \in \mathbb{N})$$

and

$$(4.20) \quad \lim_n \sum_k \bar{a}_{nk} = \alpha$$

Proof. Suppose that A satisfies the conditions (4.6), (4.7), (4.9), (4.18), (4.19), (4.20) and take any $x \in c^\lambda(\hat{F})$. Since (4.9) implies (4.5) we have by Theorem 3.6 that $\{a_{nk}\}_{k \in \mathbb{N}} \in \{c^\lambda(\hat{F})\}^\beta$

for all $n \in \mathbb{N}$ and hence Ax exists. Also from (4.9) and (4.19) we have

$$\sum_{j=0}^k |\alpha_j| \leq \sup_n \sum_j |\bar{a}_{nj}| < \infty \text{ for all } k \in \mathbb{N}.$$

This implies that $(\alpha_k) \in l_1$ and hence the series $\sum_k \alpha_k (y_k - l)$ converges where $y = (y_k) \in c$ is the sequence connected with $x = (x_k)$ by $y = \bar{F}(x)$ such that $y_k \rightarrow l$ as $k \rightarrow \infty$. By combining Lemma 3.3 with the conditions (4.9), (4.19) and (4.20) that the matrix $A = (\bar{a}_{nk})$ is in the class $(c : c)$. Now, applying the same method used in Theorem 4.5 we obtain that relation (4.12) holds and from which we obtain

$$(4.21) \quad \sum_k a_{nk} x_k = \sum_k \bar{a}_{nk} (y_k - l) + l \sum_k \bar{a}_{nk} + l a_n \quad (n \in \mathbb{N}).$$

As $n \rightarrow \infty$ from (4.21) we have,

$$A_n(x) \rightarrow \sum_k \bar{a}_{nk} (y_k - l) + l\alpha + la,$$

which shows $A(x) \in c$ and hence $A \in (c^\lambda(\hat{F}) : c)$.

Conversely suppose $A \in (c^\lambda(\hat{F}) : c)$. Since $c \subset l_\infty$, we have $A \in (c^\lambda(\hat{F}) : l_\infty)$. This leads us with Theorem 4.5 to the necessity conditions (4.6), (4.7) and (4.9). Consider the sequence $b^{(k)} = \{b_n^{(k)}\}_{n \in \mathbb{N}} \in c^\lambda(\hat{F})$ defined by (2.11) for all fixed $k \in \mathbb{N}$. Then we have $Ab^{(k)} = \{\bar{a}_{nk}\}_{n \in \mathbb{N}}$ and hence $\{\bar{a}_{nk}\}_{n \in \mathbb{N}} \in c$ for all $k \in \mathbb{N}$ which gives condition (4.19). Let $z = \sum_k b^{(k)}$. Then the linear transformation $T : c^\lambda(\hat{F}) \rightarrow c$ where $T \equiv \bar{F}$ is continuous and we obtain that $\bar{F}_n(z) = \sum_k \bar{F}_n(b^{(k)}) = 1$, $(n \in \mathbb{N})$ which gives $\bar{F}(z) = e \in c$ and hence $z \in c^\lambda(\hat{F})$.

Since $c^\lambda(\hat{F})$ and c are BK -spaces, therefore Lemma 4.1 implies the continuity of the matrix mapping $A : c^\lambda(\hat{F}) \rightarrow c$. Thus we have for every $n \in \mathbb{N}$, that

$$A_n(z) = \sum_k A_n(b^{(k)}) = \sum_k \bar{a}_{nk}.$$

This shows the necessity of (4.20).

Now, it follows that (4.9), (4.19) and (4.20) with Lemma 3.3 that $\bar{A} = (\bar{a}_{nk}) \in (c : c)$. This leads us with (4.6) and (4.7) to the consequence that the relation (4.13) holds for all $x \in c^\lambda(\hat{F})$ and $y \in c$ which are connected by $y = \bar{F}(x)$ such that $y_k \rightarrow l$ as $k \rightarrow \infty$.

Since $Ax \in c$ and $\bar{A}y \in c$, the necessity of (4.18) is obtained by (4.13). \square

Theorem 4.8. $A \in (c^\lambda(\hat{F}) : c_0)$ if and only if (4.6), (4.7), (4.9) hold and

$$(4.22) \quad \lim_n a_n = 0,$$

$$(4.23) \quad \lim_n \bar{a}_{nk} = 0 \quad (k \in \mathbb{N})$$

and

$$(4.24) \quad \lim_n \sum_k \bar{a}_{nk} = 0$$

Proof. It can be proved in the similar way as Theorem 4.7 with Lemma 4.3. \square

Theorem 4.9. $A \in (c_0^\lambda(\hat{F}) : c)$ if and only if (4.9), (4.16), (4.17) and (4.19) hold.

Proof. This result can be proved by using Lemma 3.2, Theorem 3.6 and Theorem 4.6. \square

Theorem 4.10. $A \in (c_0^\lambda(\hat{F}) : c_0)$ if and only if (4.9), (4.16), (4.17) and (4.23) hold.

Proof. This result can be proved by using Lemma 4.4, Theorem 3.6 and Theorem 4.9. \square

Lemma 4.11. [4, 5] Let X and Y be any two sequence spaces, A is an infinite matrix and B be a triangle. Then $A \in (X : Y_B)$ if and only if $BA \in (X : Y)$.

Corollary 4.12. Let $A = (a_{nk})$ be an infinite matrix and define the matrix $C = (c_{nk})$ by

$$c_{nk} = \frac{1}{\lambda_n} \sum_{i=0}^n (\lambda_i - \lambda_{i-1}) \left(\frac{f_i}{f_{i+1}} a_{ik} - \frac{f_{i+1}}{f_i} a_{i-1,k} \right); \quad (n, k \in \mathbb{N}).$$

By applying Lemma 4.11 we get, A belongs to any one of the classes $(c_0 : c_0^\lambda(\hat{F}))$, $(c : c_0^\lambda(\hat{F}))$, $(l_p : c_0^\lambda(\hat{F}))$, $(c_0 : c^\lambda(\hat{F}))$, $(c : c^\lambda(\hat{F}))$, $(l_p : c^\lambda(\hat{F}))$ if and only if the matrix C belongs respectively to the classes $(c_0 : c_0)$, $(c : c_0)$, $(l_p : c_0)$, $(c_0 : c)$, $(c : c)$, $(l_p : c)$, where $1 \leq p \leq \infty$.

5. COMPACT OPERATORS ON THE SPACES $c_0^\lambda(\hat{F})$ AND $c^\lambda(\hat{F})$

In this section, we establish some estimates for the operator norms and the Hausdorff measures of noncompactness of certain matrix operators on the spaces $c_0^\lambda(\hat{F})$ and $c^\lambda(\hat{F})$. Further, by using the Hausdorff measure of noncompactness, we characterized some classes of compact operators on these spaces.

For our investigations we need the following results.

Theorem 5.1. [15, 25] Let X and Y be FK spaces. Then $(X, Y) \subset B(X, Y)$, that is, every $A \in (X, Y)$ defines a linear operator $L_A \in B(X, Y)$ where $L_A(x) = A(x)$ and $x \in X$.

Theorem 5.2. [16] Let $X \supset \phi$ and Y be BK spaces. Then $A \in (X, l_\infty)$ if and only if $\|A\|_X^* = \sup_n \|A_n\|_X^* < \infty$. Furthermore, if $A \in (X, l_\infty)$ then it follows that $\|L_A\| = \|A\|_X^*$.

Theorem 5.3. [14] Let X be a BK space. Then $A \in (X, l_1)$ if and only if

$$\|A\|_{X,1}^* = \sup_{N \subset \mathbb{N}} \left\| \left(\sum_{n \in N} a_{nk} \right)_{k=0}^\infty \right\|_X^* < \infty, \text{ where } N \text{ is finite.}$$

Moreover, if $A \in (X, l_1)$ then $\|A\|_{X,1}^* \leq \|L_A\| \leq 4 \cdot \|A\|_{X,1}^*$.

Throughout, let $T = (t_{nk})_{n,k=0}^\infty$ be a triangle, that is $t_{nk} = 0$ for $k > n$ and $t_{nn} \neq 0$, ($n = 0, 1, \dots$) and S its inverse. The following results are known.

Theorem 5.4. [8, 25] Let $(X, \|\cdot\|)$ be a BK space. Then X_T is a BK space with $\|\cdot\|_T = \|T(\cdot)\|$.

Remark 5.5. [8] The matrix domain X_T of a normed sequence space X has basis if and only if X has a basis.

Theorem 5.6. [8] Let X be a BK space with AK and $R = S^t$, the transpose of S . If $a \in (X_T)^\beta$ then $\sum_{k=0}^\infty a_k x_k = \sum_{k=0}^\infty R_k(a) T_k(x)$ for all $x \in X_T$.

Remark 5.7. [8] The conclusion of Theorem 5.6 holds for $X = c$ and $X = l_\infty$.

Theorem 5.8. [15] Let X and Y be Banach spaces, $S_X = \{x \in X : \|x\| = 1\}$, $K_X = \{x \in X : \|x\| \leq 1\}$ and $A \in B(X, Y)$. Then the Hausdorff measure of noncompactness of a compact operator A , denoted by $\|A\|_\chi$, is given by $\|A\|_\chi = \chi(AK) = \chi(AS)$.

Furthermore, A is compact if and only if $\|A\|_\chi = 0$ (see [15]). The Hausdorff measure of noncompactness satisfies the inequality $\|A\|_\chi \leq \|A\|$ (see [15]).

Theorem 5.9. [15] Let X be a Banach space with Schauder basis $\{e_1, e_2, \dots\}$, Q be a bounded subset of X , and $P_n : X \rightarrow X$ be the projector onto the linear span of $\{e_1, e_2, \dots, e_n\}$. Then

$$\frac{1}{a} \limsup_{n \rightarrow \infty} \left(\sup_{x \in Q} \| (I - P_n)x \| \right) \leq \chi(Q) \leq \limsup_{n \rightarrow \infty} \left(\sup_{x \in Q} \| (I - P_n)x \| \right),$$

where

$$a = \limsup_{n \rightarrow \infty} \| I - P_n \|.$$

Theorem 5.10. [23] Let Q be a bounded subset of a normed space X , where X is l_p for $1 \leq p < \infty$ or c_0 . If $P_n : X \rightarrow X$ is an operator defined by $P_n(x) = (x_0, x_1, \dots, x_n, 0, 0, \dots)$, then

$$\chi(Q) = \lim_{n \rightarrow \infty} \left(\sup_{x \in Q} \| (I - P_n)x \| \right).$$

Theorem 5.11. [7] Let X be a normed sequence space and χ_T and χ denote the Hausdorff measures of noncompactness on M_{X_T} and M_X , the collection of all bounded sets in X_T and X , respectively. Then $\chi_T(Q) = \chi(T(Q))$ for all $Q \in M_{X_T}$.

Lemma 5.12. [15] Let X denote any of the spaces c_0, c or l_∞ . Then $x^\beta = l_1$ and $\|a\|_X^* = \|a\|_{l_1}$ for all $a \in l_1$.

Theorem 5.13. (a) Let $A \in (c^\lambda(\hat{F}), l_\infty)$, the matrix $\bar{A} = (\bar{a}_{nk})_{n,k=0}^\infty$ be defined by

$$\bar{a}_{nk} = \lambda_k \left[\frac{a_{nk}}{\lambda_k - \lambda_{k-1}} \cdot \frac{f_{k+1}^2}{f_k f_{k+1}} + \left(\frac{1}{\lambda_k - \lambda_{k-1}} \cdot \frac{1}{f_k f_{k+1}} - \frac{1}{\lambda_{k+1} - \lambda_k} \cdot \frac{1}{f_{k+1} f_{k+2}} \right) \sum_{j=k+1}^\infty f_{j+1}^2 a_{nj} \right]$$

for all $n, k = 0, 1, \dots$ and $\|A\|_{(c^\lambda(\hat{F}), l_\infty)} = \sup_n \left(\sum_{k=0}^\infty |\bar{a}_{nk}| \right)$.

Then

$$(5.1) \quad \|L_A\| = \|A\|_{(c^\lambda(\hat{F}), l_\infty)}.$$

(b) Let $A \in (c^\lambda(\hat{F}), l_1)$ and $\|A\|_{(c^\lambda(\hat{F}), l_1)} = \sup_{N \subset \mathbb{N}} \left(\sum_k \left| \sum_{n \in N} \bar{a}_{nk} \right| \right)$, where N is finite.

Then

$$(5.2) \quad \|A\|_{(c^\lambda(\hat{F}), l_1)} \leq \|L_A\| \leq 4 \cdot \|A\|_{(c^\lambda(\hat{F}), l_1)}.$$

Proof. (a) We assume that $A \in (c^\lambda(\hat{F}), l_\infty)$, then we have $A_n \in (c^\lambda(\hat{F}))^\beta$ for all $n = 0, 1, 2, \dots$, and it follows from the Theorem 5.6 that

$$(5.3) \quad A_n(x) = \sum_{k=0}^\infty a_{nk} x_k = \sum_{k=0}^\infty R_k(A_n) T_k(x)$$

for all $x \in c^\lambda(\hat{F})$ and $n = 0, 1, 2, \dots$, where

$$(5.4) \quad R_k(A_n) = \sum_{j=0}^\infty r_{kj} a_{nj} = \sum_{j=0}^\infty s_{jk} a_{nj}.$$

Here $T = \bar{F}$ and $S = \bar{F}^{-1}$. Therefore we have

$$R_k(A_n) = \lambda_k \left[\frac{a_{nk}}{\lambda_k - \lambda_{k-1}} \cdot \frac{f_{k+1}^2}{f_k f_{k+1}} + \left(\frac{1}{\lambda_k - \lambda_{k-1}} \cdot \frac{1}{f_k f_{k+1}} - \frac{1}{\lambda_{k+1} - \lambda_k} \cdot \frac{1}{f_{k+1} f_{k+2}} \right) \sum_{j=k+1}^\infty f_{j+1}^2 a_{nj} \right]$$

i.e.

$$(5.5) \quad R_k(A_n) = \bar{a}_{nk}$$

for all n and k . Since $c^\lambda(\hat{F})$ is a BK space, Theorem 5.2 gives

$$(5.6) \quad \|A\|_{c^\lambda(\hat{F})}^* = \sup_n \|A_n\|_{c^\lambda(\hat{F})}^* = \|L_A\|.$$

We also have $x \in S_{c^\lambda(\hat{F})}$ if and only if $y = \bar{F}(x) \in S_c$ by Theorem 5.4 and conclude from (5.3), (5.5) and the definition of the norms $\|\cdot\|_{c^\lambda(\hat{F})}^*$ and $\|\cdot\|_c^*$,

$$(5.7) \quad \|A_n\|_{c^\lambda(\hat{F})}^* = \sup \{|A_n(x)| : x \in S_{c^\lambda(\hat{F})}\} = \sup \{|\bar{A}_n(y)| : y \in S_c\} = \|\bar{A}_n\|_c^*$$

for all $n = 0, 1, 2, \dots$. By using the Lemma 5.12, we have

$$(5.8) \quad \|L_A\| = \|A\|_{c^\lambda(\hat{F})}^* = \sup_n \|A_n\|_{c^\lambda(\hat{F})}^* = \sup_n \|\bar{A}_n\|_c^* = \sup_n \left(\sum_{k=0}^{\infty} |\bar{a}_{nk}| \right) = \|A\|_{(c^\lambda(\hat{F}), l_\infty)}.$$

Part(b) is proved in exactly the same way as part(a), we apply Theorem 5.3 instead of Theorem 5.2. \square

Theorem 5.14. *Let A be an infinite matrix and put $\|A\|_{(c^\lambda(\hat{F}), l_\infty)}^{(m)} = \sup_{n > m} \sum_{k=0}^{\infty} |\bar{a}_{nk}|$.*

(a) *If $A \in (c^\lambda(\hat{F}), c_0)$, then*

$$(5.9) \quad \|L_A\|_\chi = \lim_{m \rightarrow \infty} \|A\|_{(c^\lambda(\hat{F}), l_\infty)}^{(m)}.$$

(b) *If $A \in (c^\lambda(\hat{F}), c)$, then*

$$(5.10) \quad \frac{1}{2} \cdot \lim_{m \rightarrow \infty} \|A\|_{(c^\lambda(\hat{F}), l_\infty)}^{(m)} \leq \|L_A\|_\chi \leq \lim_{m \rightarrow \infty} \|A\|_{(c^\lambda(\hat{F}), l_\infty)}^{(m)}.$$

(c) *If $A \in (c^\lambda(\hat{F}), l_\infty)$, then*

$$(5.11) \quad 0 \leq \|L_A\|_\chi \leq \lim_{m \rightarrow \infty} \|A\|_{(c^\lambda(\hat{F}), l_\infty)}^{(m)}.$$

Proof. Let us assume that limits in (5.9) to (5.10) exist. We write $K = \{x \in c^\lambda(\hat{F}) : \|x\| \leq 1\}$

(a) Applying Theorem 5.10, we have

$$(5.12) \quad \|L_A\|_\chi = \chi(AK) = \lim_{m \rightarrow \infty} \left\{ \sup_{x \in K} \|(I - P_m)Ax\| \right\},$$

where $P_m : c_0 \rightarrow c_0$ ($m = 0, 1, 2, \dots$) is the projector such that $P_m = (x_0, x_1, \dots, x_m, 0, 0, \dots)$ for $x = (x_k) \in c_0$. It is known that $\|I - P_m\| = 1$ for all m . Let $A_m = (\acute{a}_{nk})$ be the infinite matrix with

$$\acute{a}_{nk} = \begin{cases} 0 & , 0 \leq n \leq m \\ \bar{a}_{nk} & , m < n. \end{cases}$$

We have

$$(5.13) \quad \sup_{x \in K} \|(I - P_m)Ax\| = \|L_{A_m}\| = \|A_m\|_{(c^\lambda(\hat{F}), l_\infty)} = \|A\|_{(c^\lambda(\hat{F}), l_\infty)}^{(m)},$$

and we obtain (5.9) from (5.12) and (5.13).

(b) This proof can be done exactly in the same way as in (a). If $P_m : c \rightarrow c$ ($m = 0, 1, 2, \dots$) is the projector such that $P_m(x) = le + \sum_{k=0}^m (x_k - l)e^{(k)}$ then $\|I - P_m\| = 2$ for all m . By applying Theorem 5.9 we can prove part (b).

(c) Let $P_m : l_\infty \rightarrow l_\infty$ ($m = 0, 1, 2, \dots$) by $P_m(x) = (x_0, x_1, \dots, x_m, \dots)$ for $x = (x_k) \in l_\infty$.

Since $AK \subset P_m(AK) + (I - P_m)(AK)$, by applying properties of χ , we obtain
 $\chi(AK) \leq \chi(P_m(AK)) + \chi((I - P_m)(AK)) = \chi((I - P_m)(AK)) \leq \sup_{x \in K} \|(I - P_m)Ax\|$
 i.e. $0 \leq \chi(AK) \leq \|L_{A(m)}\| = \lim_{m \rightarrow \infty} \|A\|_{(c^\lambda(\hat{F}), l_\infty)}^{(m)}$. \square

Corollary 5.15. *If either $A \in (c^\lambda(\hat{F}), c)$ or $A \in (c^\lambda(\hat{F}), c_0)$, then L_A is compact if and only if*

$$(5.14) \quad \lim_{m \rightarrow \infty} \|A\|_{(c^\lambda(\hat{F}), l_\infty)}^{(m)} = 0.$$

If $A \in (c^\lambda(\hat{F}), l_\infty)$, then L_A is compact if the condition (5.14) holds.

Theorem 5.16. *Let A be an infinite matrix and put*

$$\|A\|_{(c^\lambda(\hat{F}), l_1)}^{(m)} = \sup_{N \subseteq \mathbb{N} \setminus \{0, 1, \dots, m\}} \left(\sum_k \left| \sum_{n \in N} \bar{a}_{nk} \right| \right),$$

N is finite. Then

$$\|L_A\|_\chi = \lim_{m \rightarrow \infty} \|A\|_{(c^\lambda(\hat{F}), l_1)}^{(m)}.$$

Proof. The proof is similar to that of Theorem 5.14. \square

Corollary 5.17. *If $A \in (c^\lambda(\hat{F}), l_1)$, then L_A is compact if and only if*

$$\lim_{m \rightarrow \infty} \|A\|_{(c^\lambda(\hat{F}), l_1)}^{(m)} = 0.$$

Corollary 5.18. (a) *If $A \in (c^\lambda(\hat{F}), bv)$ then L_A is compact if and only if*

$$\lim_{m \rightarrow \infty} \sup_{N \subseteq \mathbb{N} \setminus \{0, 1, \dots, m\}} \left(\sum_k \left| \sum_{n \in N} (\bar{a}_{nk} - \bar{a}_{n-1, k}) \right| \right) = 0.$$

Remark 5.19. *The previous results would be true for $c_0^\lambda(\hat{F})$ instead of $c^\lambda(\hat{F})$.*

REFERENCES

- [1] Abdullah Alotaibi, M. Mursaleen, Badriah AS Alamri and S. A. Mohiuddine, Compact operators on some Fibonacci difference sequence spaces, J. Ineq. Appl. 2015, 2015:203.
- [2] B. Altay, F. Başar, Certain topological properties and duals of the domain of a triangle matrix in a sequence space, J. Math. Anal. Appl. 336(2007) 632-645
- [3] J. Banaś, K. Goebel, Measure of noncompactness in Banach spaces, Lecture Notes in Pure and Applied Mathematics, Vol. 60, Marcel Dekker, New York Basel, 1980.
- [4] F. Başar, B. Altay, M. Mursaleen, Some generalizations of the space bv_p of p -bounded variation sequences. Nonlinear Anal. TMA 68(2008) 273-287.
- [5] F. Başar, B. Altay, On the space of sequences of p -bounded variation and related matrix mappings, Ukrainian Math. J. 55(1)(2003) 136-147.
- [6] Anupam Das and B. Hazarika, Some properties of Generalized Fibonacci difference bounded and p -absolutely convergent sequences, arXiv:1604.00182v1.
- [7] I. Djolović, Compact operators on the spaces $a_0^r(\Delta)$ and $a_c^r(\Delta)$, J. Math. Anal. Appl. 340(1)(2008) 291-303.
- [8] A.M. Jarrah, E. Malkowsky, Ordinary, absolute and strong summability and matrix transformations, Filomat 17(2003) 59-78.
- [9] P.K. Kamthan, M. Gupta, Sequence Spaces and Series, Marcel Dekker Inc., New York and Basel, 1981.
- [10] EV Kara, Some topological and geometrical properties of new Banach sequence spaces, J. Ineq. Appl. 2013, 2013:38
- [11] M. Kirişçi, F. Başar, Some new sequence spaces derived by the domain of generalized difference matrix. Comput. Math. Appl. 60(2010) 1229-1309.
- [12] H. Kizmaz, On certain sequence spaces, Canad. Math. Bull. 24(2)(1981) 169-176.
- [13] T. Koshy, Fibonacci and Lucas Numbers with applications, Wiley, 2001.

- [14] E. Malkowsky, Klassen von Matrixabbildungen in paranormierten FK-Räumen, *Analysis (Munich)* 7 (1987) 275-292.
- [15] E. Malkowsky, V. Rakočević, An introduction into the theory of sequence spaces and measure of noncompactness, in: *Zb. Rad. (Beogr.)*, vol. 9(17), Matematički institut SANU, Belgrade, 2000, pp. 143-234.
- [16] E. Malkowsky, V. Rakočević, S. Živković, Matrix transformations between the sequence spaces $w_0^p(\Lambda)$, $v_0^p(\Lambda)$, $c_0^p(\Lambda)$, $1 < p < \infty$, and BK spaces, *Appl. Math. Comput.* 147 (2004) 377-396.
- [17] E. Malkowsky, Measure of noncompactness and applications, *Contem. Anal. Appl. Math.* 1(1)2013 2-19.
- [18] M. Mursaleen, Generalized spaces of difference sequences. *J. Math. Anal. Appl.* 203(3)(1996) 738-745.
- [19] M. Mursaleen, V. Karakaya, H. Polat and N. Simsek, Measure of noncompactness of matrix operators on some difference sequence spaces of weighted means, *Comput. Math. Appl.* 62(2011) 814-820.
- [20] M. Mursaleen, AK Gaur, AH Saifi, Some new sequence spaces and their duals and matrix transformations. *Bull. Calcutta Math. Soc.* 88(3)(1996) 207-212.
- [21] M. Mursaleen, AK Noman, Compactness by the Hausdorff measure of noncompactness, *Nonlinear Anal. TMA*, 73(8)(2010) 2541-2557.
- [22] M. Mursaleen, AK Noman, On the spaces of λ -convergent and bounded sequences, *Thai J. Math.* 8 (2)(2010) 311-329.
- [23] V. Rakočević, Measures of noncompactness and some applications, *Filomat*, 12(2)(1998) 87-120.
- [24] Michael Stieglitz, Hubert Tietz, Matrixtransformationen von Folgenräumen Eine Ergebnisübersicht, *Math. Z.* 154(1977) 1-16.
- [25] A. Wilansky, *Sumability Through Functional Analysis*. North-Holland Mathematics Studies, vol. 85. Elsevier Amsterdam. (1984)